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**NOTE**

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# Singular behaviour of the eigenvector component of the partially chaotic ensemble

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**Abstract** : The behaviour of the eigenvector component of a band matrix in three dimensions is studied. It is shown that for small values of the eigenvector component it has a logarithmic divergence. This shows that in addition to spacing distribution, one could also use eigenvector distribution to study chaotic systems.

**Keywords** : Matrix ensembles, chaos

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Recently, there has been much interest in the study of chaos in Quantum systems. In the past, most of the work has been concentrated [1] on studying chaos using the distribution of the nearest level spacing. If this distribution turns out to be Wigner distribution [2], then the system is chaotic while if it is Poisson distribution, then the system is integrable. So far no analytic studies have been carried out for the eigenvector component distribution of such systems. The eigenvector component distribution if found to be different than what one finds for *GOE* will provide additional information about the chaotic behaviour of the system. In the present work, we would like to show how the behaviour of the eigenvector component changes if one uses non-invariant Hamiltonian matrix elements distribution. For this purpose, we shall consider $3 \times 3$ band matrices [3].

Here, we shall derive a general expression for the distribution of the eigenvector component of a $3 \times 3$ real symmetric Hamiltonian matrix. Let $T_{ij}$ $(i, j = 1,\ldots,3)$ be the elements of the orthogonal matrix $T$ which is connected with the Hamiltonian matrix $H$ and its diagonal form $E$ by

$$H = T E \tilde{T}, \tag{1}$$

where ~ denotes transpose of $T$.

According to the random matrix theory [4], the joint distribution of the elements $T_{ij}$ denoted by $P(\{T_{ij}\})$ is given by

$$P(\{T_{ij}\}) = \int f(\{H_{ij}\}) \prod_{i<j} |E_i - E_j|$$

$$\prod_{i=1}^{3} \delta\left(\sum_j T_{ij}^2 - 1\right) \prod_{i<k} \delta\left(\sum_j T_{ij} T_{kj}\right) \prod_{i=1}^{3} dE_i, \tag{2}$$

where $f(\{H_{ij}\})$ is the distribution of the matrix elements $H_{ii}$ $(i = 1, 2, 3)$ and $H_{ij} (i < j)$.

Making the orthogonal transformation

$$u = \frac{1}{\sqrt{2}} (E_2 - E_3),$$

and $$v = \frac{1}{\sqrt{2}} (E_2 + E_3)$$

and using the orthonormality conditions of $T_{ij}$,

$$\sum_{j=1}^{3} T_{ij}^2 = 1, \qquad i = 1, 2, 3;$$

$$\sum_{j=1} T_{ij} T_{kj} = 0, \quad i \neq k$$

and carrying out integrations over all $T_{ij}$'s except $T_{11}$, one gets the following expression for the probability density function of $T_{11}$,

$$P(T_{11}) = \int f|u|\left|q^2 - \frac{u^2}{2}\right| du\, dv\, d\phi\, d\gamma, \tag{3}$$

where $f$ is now a function of $u$, $q$, $v$, $\phi$ and $\gamma$.

$$\begin{aligned}
& f(H_{11}, H_{22}, H_{33}, H_{12}, H_{13}, H_{23}) \\
&= f\left[\left[\left(q + \frac{u}{\sqrt{2}}\right)T_{11}^2 + \frac{v}{\sqrt{2}} + \frac{u}{\sqrt{2}}\left[2\left(1 - T_{11}^2\right)\sin^2\gamma - 1\right]\right],\right. \\
& \left[\left(q + \frac{u}{\sqrt{2}}\right)\left(1 - T_{11}^2\right)\cos^2\phi + \frac{v}{\sqrt{2}} + \frac{u}{\sqrt{2}}\right. \\
& \qquad \left.\left[2(\sin\phi\cos\gamma - T_{11}\cos\phi\sin\gamma)^2 - 1\right]\right], \\
& \left[\left(q + \frac{u}{\sqrt{2}}\right)\left(1 - T_{11}^2\right)\sin^2\phi + \frac{v}{\sqrt{2}} + u\sqrt{2}\right. \\
& \left.\left[2(\cos\phi\cos\gamma - T_{11}\sin\phi\sin\gamma)^2 - 1\right]\right], \\
& \qquad \left[\left(q + \frac{u}{\sqrt{2}}\right)T_{11}\sqrt{1 - T_{11}^2}\cos\phi\right. \\
& \left. + \frac{u}{\sqrt{2}} 2\sqrt{1 - T_{11}^2}\sin\gamma(-\sin\phi\cos r - T_{11}\cos\phi\sin\gamma)\right], \\
& \left[\left(q + \frac{u}{\sqrt{2}}\right)T_{11}\sqrt{1 - T_{11}^2}\sin\phi + \frac{u}{\sqrt{2}}2\sqrt{1 - T_{11}^2}\right. \\
& \qquad \left.\sin\gamma(\cos\phi\cos\gamma - T_{11}\sin\phi\sin\gamma)\right], \\
& \left[\left(q + \frac{u}{\sqrt{2}}\right)\left(1 - T_{11}^2\right)\cos\phi\sin\phi + \frac{u}{\sqrt{2}}\right. \\
& \qquad 2(-\sin\phi\cos\gamma - T_{11}\cos\phi\sin\gamma) \\
& \qquad \left.\left.(\cos\phi\cos\gamma - T_{11}\sin\phi\sin\gamma)\right]\right]
\end{aligned} \tag{4}$$

We shall now use expression (3) for a given distribution of $f$. If $f$ is rotationally invariant *GOE* distribution given by

$$f = \exp(-TrH^2), \tag{5}$$

then in terms of the variables of expression (3), it is given by

$$f = \exp\left[-\left(u^2 + \frac{3}{2}v^2 + q^2 + \sqrt{2}vq\right)\right], \tag{6}$$

and we immediately see from expression (3) that for a rotationally invariant distribution like (5), $P(T_{11})$ is given by

$$P(T_{11}) = \text{constant}, \tag{7}$$

where the constant can be fixed by the normalization condition

$$\int_{-1}^{1} P(T_{11})dT_{11} = 1. \tag{8}$$

This is in agreement with the earlier result [4] of the eigenvector component distribution which gives the well known Porter-Thomas distribution for the width of the compound nucleus resonance in three dimensions

We now consider band matrices [3] for which $f$ is given by

$$f = [\exp(-\mathrm{Tr}H^2)]\,\delta(H_{13}). \tag{9}$$

Using expressions (3), (4) and (9) and after some simplification we get $P(T_{11})$ for band matrices to be given by

$$\begin{aligned}
P(T_{11}) &= \frac{1}{\sqrt{1 - T_{11}^2}}\int \exp\left(-\frac{2}{3}q^2 - u^2\right)|u|\left|q^2 - \frac{u^2}{2}\right| \\
&\times\left[\frac{u^2}{2}\cos^2\phi + T_{11}^2\left(\frac{u^2}{2} - q^2\right)\sin^2\phi\right]^{-\frac{1}{2}} dq\, du\, d\phi.
\end{aligned} \tag{10}$$

Writing the integral over $u$ as $\sqrt{2}\,q$ to $\infty$ and 0 to $\sqrt{2}\,q$ and carrying out integration over $\phi$ we get

$$\begin{aligned}
P(T_{11}) = \frac{1}{\sqrt{1 - T_{11}^2}} & \left[\int_0^{\infty} dq \int_q^{\infty} du \exp\left(-\frac{2}{3}q^2 - u^2\right)\left(\frac{u^2}{2} - q^2\right)\right. \\
& \times {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\frac{u^2}{2} - T_{11}^2\left(\frac{u^2}{2} - q^2\right)}{\frac{u^2}{2}}\right) \\
& + \int_0^{\infty} dq \int_0^{q} du \exp\left(-\frac{2}{3}q^2 - u^2\right)\frac{u\left(q^2 - \frac{u^2}{2}\right)}{\sqrt{\frac{u^2}{2}\left(1 - T_{11}^2\right) + q^2 T_{11}^2}} \\
& \left.\times {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\frac{u^2}{2}}{\frac{u^2}{2}\left(1 - T_{11}^2\right) + q^2 T_{11}^2}\right)\right],
\end{aligned} \tag{11}$$

where ${}_2F_1(a, b; c; x)$ is a hypergeometric function [5]

We are now ready to discuss the behaviour of $P(T_{11})$ as $T_{11} \to 0$. From the properties of hypergeometric functions [5], we see that the hypergeometric functions ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right)$ in expression (12) have a logarithmic singularity when $z \to 1$. This gives the behaviour of $P(T_{11})$ as $T_{11} \to 0$ as

$$P(T_{11}) = (\text{const.})\left[-\ln T_{11}^2 + 4\ln 2\right].$$

Thus the probability density function of the eigenvector component of a band matrix has singular behaviour for small values of the eigenvector component which is very different than the behaviour of $P(T_{11})$ for Gaussian orthogonal Ensemble (GoE) which has no such singularity as $T_{11} \to 0$

The first remark which we would like to add is that the logarithmic behaviour even though proved analytically for

$3 \times 3$ band matrix should be there for a general $n \times n$ band matrix. This is based on the fact that many results which were first proved for small dimensional matrices, turned out to be there for the general case also. A well known example is Wigner's spacing distribution.

As one can see expression (4) for the distribution of $T_{11}$ is quite general and can be used for other non-invariant trace ensembles also. The important new result which has been established here is that the behaviour of the probability density function of the eigenvector component for small values of eigenvector component can also be used to study chaos just like the behaviour of the nearest level spacing distribution for small value of the spacing.

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